Analytic Mechanics: Discussion Worksheet 2

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This worksheet is primarily concerned with transformations and rotations. The point today is to become so doggone comfortable with the philosophy, mathematics, and conventions of rotations (and transformations in general) that you never have to think about it again. We'll fail, of course, but we'll give it a good try!

1 Define, in a sentence or two or three, the following terms. Try to be *insightful* and *instructive* as opposed to *precise*; this is for understanding, not for a grade.

1.1 Rotation

1.2 Space point

1.3 Space axes

1.4 Body axes

1.5 Passive rotation

1.6 Active rotation

1.7 Line integral

1.8 Conservative force

1.9 Field (physics, not math)

1.10 Force field

1.11 Pseudovector (and why does it behave that way?)

2 Rotation Conventions

I'm in a spaceship near earth, and I want a rotation matrix to describe the earth's rotational motion over a period of 6 hours (i.e., the discrete transformation from noon to 6 PM, not every time in between). Let $\hat{e_3}$ point from the south pole to the north pole, and let $\hat{e_2}$ point from the center of the earth to the sun.

2.1 Find the rotation matrix $\hat{\Lambda}$.

2.2 Is this an active or passive rotation?

2.3 Now, let $\hat{e_3}$ point in a direction halfway between the pole-pole line and the center-sun line. Find $\hat{\Lambda}$ either explicitly or as a product of matrices.

3 Is there any reason why I couldn't define the origin of the body frame to have the space coordinates $\vec{O'} = (1, 0, 0)$? Why is this acceptable, or why is it not?

4 Give 3 rotation matrices that will move the north pole of a sphere to its south pole. Would this be an active or passive rotation, in your opinion? Note: at least one of your matrices should have a different determinant from the others.

5 Euler-like Rotations

You are looking at a unit sphere in space. You want to rotate your point of view (what kind of rotation is this?) so that the point on the sphere at $(0, \sqrt{2}, \sqrt{2})$ goes to $(\sqrt{2}, \sqrt{2}, 0)$ and $(0, 0, 1)$ goes to $(0, 1, 0)$.

5.1 Draw a picture of this rotation using your best 3D drawing skills.

5.2 Describe two ways of building this rotation R out of 3 simple rotations, one of which should be according to the Euler convention.

5.3 Calculate the rotation matrix $\hat{R}$ both ways, and ensure that they agree.

5.4 Are any points left unmoved by this transformation? Why or why not? If you choose "yes," describe them.

5.5 How would you calculate such points if they existed?

6 Hyperbolic transformations (optional)

The exponential of a matrix, $e^{\hat{M}}$, is defined by the power series of the exponential. This includes only powers of the argument to the exponential, and powers of a matrix are well defined.

6.1 Show that all rotations in 2 dimensions can be written in the following form:

$$\begin{aligned} \hat{R} &= e^{\theta\hat{\sigma}} \\ \hat{\sigma} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

6.2 Now, consider another transformation just like the previous one, but where the exponentiated matrix (the "generator" of the transformations) is given by:

$$\hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In this case, what is the form of the transformation matrix?

6.3 Is the norm of a vector $\hat{r}$ preserved? (That is, $\hat{r}\cdot\hat{r}$.) What quantity *is* preserved?

6.4 Does your answer to the previous question give you a hint as to how to combine two vectors to form an *invariant* quantity – that is, how to redefine the dot product to be invariant under this transformation rule?